On *A*-eccentric Graphs

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Abstract

Motivated by the concept called unique eccentric point (u.e.p) [14], Kishori et al. in [2] generalized the concept as *k*- eccentric point graph. A graph *tt* is called an *unique eccentric point (u.e.p) graph* if each point of *tt* has a unique eccentric point where as in *k-eccentric point graph* every vertex has exactly *k*-eccentric vertices. Here we are denoting *k*-eccentric graph as *A*-eccentric graph and studied its property with peripheral path matrix.

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1 Introduction

Let tt be a connected, nontrivial graph with vertex set $V(tt)$ and edge set $E(tt)$ and let $|V(tt)| = n$ and $|E(tt)| = m$. Let u and v be two vertices of a graph tt. The *distance* $d(u, v|tt)$ between the vertices u and v is the length of a shortest path connecting u and v. If $u = v$ then $d(u, v|tt) = 0$. The *eccentricity* $e(v)$ of a vertex v in a graph tt is the distance between v and a vertex farthest from v in tt. The *diameter* $diam(tt)$ of tt is the maximum eccentricity of tt, while the *radius* $rad(tt)$ is the smallest eccentricity of tt. A vertex v with $e(v) = diam(tt)$ is called a *peripheral* vertex of tt. A set of peripheral vertices of tt is called as periphery and is denoted as $P(tt)$. The *peripheral path matrix* $M_p(tt) = [p_{ij}]$, whose entries are 1 if there is a peripheral path between v_i and v_j in tt and 0 otherwise.

The peripheral path energy (p-energy (in short)) of a graph tt is defined as the sum of the absolute values of p- eigenvalues of the $M_p(tt)$. i.e,

$$E_p = E_p(tt) = \sum_{i=1} |\alpha_i| \tag{1}$$

The form of Eq. (1) is chosen so as to be fully analogous to the definition of graph energy [8, 5, 7].

$$E = E(tt) = \sum_{i=1} |\lambda_i|$$

where, $\lambda_1, \lambda_2, ..., \lambda_n$ are the ordinary eigenvalues [9], i.e the eigenvalues of the adjacency matrix $A(tt)$. Observe that the graph energy $E(tt)$ in a past few years has been extensively studied and surveyed in Mathematics and Chemistry [12, 13, 15, 17, 18, 19, 21, 25, 26, 10, 11, 16, 20, 23]. Through out the paper $|P(tt)| = k$ with labellings $v_1, v_2, ..., v_k$, where $2 \leq k \leq n$.

The *characteristic polynomial* of $M_p(tt)$ is the $det(\alpha I \quad M_p(tt))$. It is referred to as a characteristic polynomial of tt and is denoted by $\psi(tt; \alpha) = c_0\alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + \ldots + c_n$. The roots $\alpha_1, \alpha_2, ..., \alpha_n$ of the polynomial $\psi(tt; \alpha)$ is called the *eigenvalues* of $M_p(tt)$.

The eigenvalues of $M_p(tt)$ are said to be the *peripheral path eigenvalues* (or p-eigenvalues (in short)) of tt. Since $M_p(tt)$ is a real, symmetric matrix, the p-eigenvalues are real and can be ordered in non-increasing order, $\alpha_1 \geq \alpha_2 \geq, ..., \geq \alpha_n$. Then the *p-spectrum* of a graph tt is the set of p-eigenvalues of $M_p(tt)$, together with the multiplicities as p-eigenvalues of $M_p(tt)$. If the p-eigenvalues of $M_p(tt)$ are $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$ and their multiplicities are $m(\alpha_1), m(\alpha_2), ..., m(\alpha_n)$, then the result will be

$$p - Spec(tt) = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ m(\alpha_1) & m(\alpha_2) & \ldots & m(\alpha_n) \end{pmatrix}.$$

For Example, let tt be a graph as shown below Figure 1.

tt :

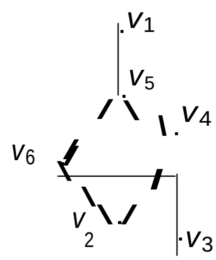

Figure 1: tt is a graph of order $n = 6$ with $k = 3$ peripheral vertices.

$$M_p(tt) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}_{6\times 6}$$

Let $\varepsilon_v = \{u : d(u, v) = diam(tt)\}$. Clearly, $|\varepsilon_{v1}| = 2$ and $|\varepsilon_{v2}| = |\varepsilon_{v3}| = 1$ and characteristic polynomial of tt is $\psi(tt; \alpha) = -2\alpha^4 + \alpha^6$ whose, p- eigenvalues are $-\sqrt{2}, \sqrt{2}, 0, 0, 0, 0$. And hence, p-energy of tt is 2.8284.

Motivated by the concept called unique eccentric point (u.e.p) [14], Kishori et al. in [2] generalized the concept as k- eccentric point graph. A graph tt is called an *unique eccentric point (u.e.p) graph* if each point of tt has a unique eccentric point where as in *k-eccentric point graph* every vertex has exactly k-eccentric vertices. Here we are denoting k-eccentric graph as A-eccentric graph and studied its property with peripheral path matrix. For more details about the Peripheral path matrix, Peripheral path energy, Peripheral distance energy, Peripheral path equi-energy and Peripheral Wiener index one can refer [22],

Deftnition 1.1. *A vertex u is said to be an eccentric vertex of v if $e(v) = d(u, v)$ $u, v \in V(tt)$. A graph tt is A- eccentric graph if for every x of $V(tt)$, there are A- eccentric vertices.*

For Example: Refer the Figure 2.

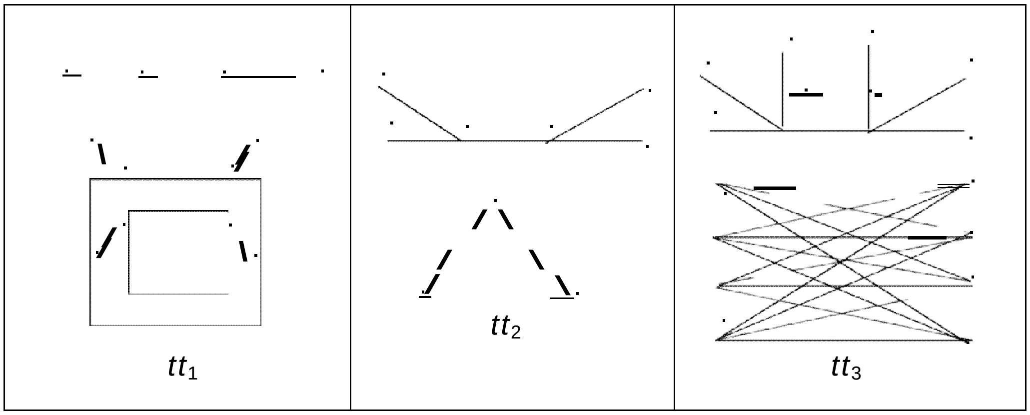

Figure 2: tt_1, tt_2, and tt_3 are the 1, 2 and 3 eccentric graphs respectively

2 Properties of *A*-eccentric graph

Theorem 2.1. *Let tt be A-eccentric graph. Then,*

1. *A is an p-eigenvalues of tt.*

2.If tt is connected, then the multiplicity of A is 1.

3.For any p-eigenvalue α of tt, we have $|\alpha| \leq A$

Proof.

1. Let $u = [1, 1, 1..., 1]^t$, then if M_p is peripheral path matrix of tt, we have,

$$M_p = Au. \tag{2}$$

This is true because there are A 1^{j} in each row. Thus A is a p-eigenvalue of tt.

2. Let $X = [x_1, x_2, ..., x_n]^t$ denote any non zero vector for which

$$AX = AX \tag{3}$$

suppose that x_j is an entry of X with the largest absolute value, then the Eq (3) can be expressed as

$$(Ax)_j = Ax_j$$

and hence

$$\sum x_i = Ax_j$$

where summation runs over those A vertices v_i which are an eccentric vertex to v_j. If tt is connected, we may proceed successively in this way, eventually showing that all entries of X are equal. Thus X is a multiple of u and the space of eigenvector associated with the p-eigenvalue A has dimension 1.

3. Suppose that

$$Ay = \alpha y \ , \ y \neq 0 \tag{4}$$

and y_j denote an entry of y which is least in absolute value, as in condition 2. we have,

$$\sum y_i = \alpha y_j \text{ and}$$

$$|\alpha||y_j| = |\sum y_i|$$

$$|\alpha||y_j| \leq \sum |y_i|$$

$$|\alpha||y_j| = A|y_i|$$

$$\Longrightarrow |\alpha| \leq A \text{ as required.}$$

□

Proposition2.2. *If tt is A-eccentric graph and α is a p-eigenvalue of $M_p(tt)$, then no p-eigenvalue of $M_p(tt)$ has absolute value greater than A.*

Proof. Let α be a p-eigenvalue of $M_p(tt)$ and $x = [x_1, x_2, ..., x_k]^t$ be corresponding eigenvector. Let x_i be the entry of x whose absolute value is greater. Hence,

$$\alpha x_i = \sum_{j=1}^{k} a_{ij} x_j \ ; \ i = 1, 2, ...k.$$

Now

$$|\alpha||x_i| \leq |x_i| \sum_{j=1}^{k} a_{ij} \ ; \ i = 1, 2, ...k.$$

since tt is $A-$ eccentric graph,

$$\sum_{j=1}^{k} a_{ij} = A \ ; \ i = 1, 2, ...k.$$

Hence,

$$|\alpha||x_i| \leq |x_i|.|A| \Longrightarrow |\alpha| \leq |A|.$$

□

Corollary2.3. *Suppose tt is unique eccentric point graph then no p-eigenvalue of $M_p(tt)$ has absolute value greater than* 1.

Proof. Let α be a p-eigenvalue of $M_P(tt)$ and $x = [x_1, x_2, ...x_k]^t$ corresponding eigenvector. Let x_i be the entry of x whose absolute value is greatest. Hence,

$$\alpha x_i = \sum_{j=1}^{k} a_{ij} x_j \; ; \; i = 1, 2, ...k.$$

Now.

$$|\alpha||x_i| \leq |x_i| \sum_{j=1}^{k} a_{ij} \; ; \; i = 1, 2, ...k.$$

since tt is unique eccentric point graph,

$$\Longrightarrow |\alpha| \leq 1.$$

□

Proposition2.4. *Let $M_p(tt)$ be $n \times n$ real matrix with $v_1, v_2, ..., v_k$ peripheral vertices. Let $M_{p1}(tt)$ be $k \times k$ real sub matrix of $M_p(tt)$. If tt is unique eccentric point graph then there is a non-zero column vector x such that $Ax = x$*

Proof. Let $M_{p1}(tt) - I$ be a matrix which has the properties that the sum of the entries in each column is equal to zero. Since tt is unique eccentric point graph, $M_{p1}(tt) - I$ results in to a matrix whose first row is zero. (If not a small operation can be applied, i.e. $R_1 \rightarrow R_1 + R_2 +, ..., +R_k$ to $M_{p1}(tt) - I$). Hence, $|M_{p1}(tt) - I| = 0$. Thus 1 is a p-eigenvalue of A and there is an eigenvector $x \neq 0$ such that $Ax = x$.□

3 *A*-Eccentric Trees:

For A-eccentric tree, the peripheral path matrix $M_p(T)$ is as follows:

$$M_p(T) = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_\ell \\ v_{\ell+1} \\ v_{\ell+2} \\ \vdots \\ v_{2\ell} \\ v_{2\ell+1} \\ v_{2\ell+2} \\ \vdots \\ v_n \end{array} \begin{array}{cccccccccccc} v_1 & v_2 & \cdots & v_\ell & v_{\ell+1} & v_{\ell+2} & \cdots & v_{2\ell} & v_{2\ell+1} & v_{2\ell+2} & \cdots & v_n \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}_{n\times n}$$

This matrix representation takes the form

$$M_p(T) = \left[\begin{array}{c|c} A_{2A\times 2A} & B_{2A\times(n-2A)} \\ \hline B^t_{(n-2A)\times 2A} & C_{(n-2A)\times n-2A)} \end{array}\right]_{n\times n}.$$

Note that sub matrix B, B^t and C are zero - matrix, where as A is non-zero matrix. A sub matrix A is $2A \times 2A$ sub matrix of $M_p(T)$, whose entries are as follows:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Symbolically A can be represented as

$$A = \left[\begin{array}{c|c} 0_{A\times A} & J_{A\times A} \\ \hline J^t_{A\times A} & 0_{A\times A} \end{array}\right]_{2A\times 2A}$$

Where J is a $A \times A$ matrix having all entries 1.

Observation3.1. *A sub matrix A of $M_p(T)$ has just two linearly independent rows and so its rank is 2. Consequently 0 is an p-eigenvalue of $M_p(T)$ with multiplicity $2A-2$.*

Next, we give the Characteristic polynomial of a sub matrix A of $M_p(T)$. Suppose,

$$A = \left[\begin{array}{cccc|cccc}
a_{11} & \cdots & a_{1,(A-1)} & a_{1,A} & a_{1,(A+1)} & \cdots & a_{1,2(A-1)} & a_{1,2A} \\
a_{21} & \cdots & a_{2,(A-1)} & a_{2,A} & a_{2,(A+1)} & \cdots & a_{2,2(A-1)} & a_{2,2A} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{(A-1),1} & \cdots & a_{(A-1),(A-1)} & a_{(A-1),A} & a_{(A-1),(A+1)} & \cdots & a_{(A-1),2(A-1)} & a_{(A-1),2A} \\
a_{A,1} & \cdots & a_{A,(A-1)} & a_{A,A} & a_{A,(A+1)} & \cdots & a_{A,2(A-1)} & a_{A,2A} \\
\hline
a_{(A+1),1} & \cdots & a_{(A+1),(A-1)} & a_{(A+1),A} & a_{(A+1),(A+1)} & \cdots & a_{(A+1),2(A-1)} & a_{(A+1),2A} \\
a_{(A+2),1} & \cdots & a_{(A+2),(A-1)} & a_{(A+2),A} & a_{(A+2),(A+1)} & \cdots & a_{(A+2),2(A-1)} & a_{(A+2),2A} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{(2A-1),1} & \cdots & a_{(2A-1),(A-1)} & a_{(2A-1),A} & a_{(2A-1),(A+1)} & \cdots & a_{(2A-1),2(A-1)} & a_{(2A-1),(2A)} \\
a_{(2A),1} & \cdots & a_{(2A),(A-1)} & a_{(2A),A} & a_{(2A),(A+1)} & \cdots & a_{(2A),2(A-1)} & a_{(2A),(2A)}
\end{array}\right]_{2A\times 2A}$$

$$\Longrightarrow A = \left[\begin{array}{ccccc|ccccc}
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
\hline
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0
\end{array}\right]_{2A\times 2A}$$

then,

$$|A - \alpha I| = \left|\begin{array}{cccccccccc}
-\alpha & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & -\alpha & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\alpha & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & -\alpha & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & -\alpha & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 1 & 1 & 0 & -\alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & -\alpha & 0 \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & -\alpha
\end{array}\right|_{2A\times 2A} = 0$$

now, subtract $(A+1)^{th}$ row from $(A+2)^{th}$, $(A+3)^{th}$, ..., $(2A-1)^{th}$, $(2A)^{th}$ rows, then the result will be,

$$\begin{vmatrix} -\alpha & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -\alpha & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & -\alpha & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -\alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \alpha & -\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \alpha & 0 & \cdots & -\alpha & 0 \\ 0 & 0 & \cdots & 0 & 0 & \alpha & 0 & \cdots & 0 & -\alpha \end{vmatrix}_{2A\times 2A} = 0$$

Adding $(A + 2)^{th}$, $(A + 3)^{th}$, ..., $(2A - 1)^{th}$, $(2A)^{th}$ column to $(A + 1)^{th}$ column,

$$\begin{vmatrix} -\alpha & 0 & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\ 0 & -\alpha & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha & 0 & A & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & -\alpha & A & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -\alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{vmatrix}_{2A\times 2A} = 0$$

Again by subtracting $(1)^{st}$ column from $(2)^{nd}$, $(3)^{rd}$, ..., $(A)^{th}$ column, we have

$$\begin{matrix} -\alpha & \alpha & \cdots & \alpha & \alpha & A & 1 & \cdots & 1 & 1 \\ 0 & -\alpha & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & & & & \vdots \end{matrix}$$

$$\begin{vmatrix} \vdots & & & & & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & -\alpha & 0 & A & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & -\alpha & A & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & -\alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{vmatrix}_{2A\times 2A} = 0$$

Expanding from 1st column.

$$\Longrightarrow (-1)^{1+1}A_1 + (-1)^{A+1+1}A_2 = 0 \qquad (5)$$

$$Let\ A_1 = \begin{pmatrix} -\alpha & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\alpha & -\alpha & A & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & -\alpha & 0 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ 0 & \cdots & & & & & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{pmatrix}_{(2A-1)\times(2A-1)}$$

$$Let\ A_2 = \begin{pmatrix} \alpha & \cdots & \alpha & \alpha & A & 1 & \cdots & 1 & 1 \\ -\alpha & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\alpha & 0 & A & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & -\alpha & A & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{pmatrix}_{(2A-1)\times(2A-1)}$$

now, the determinant of $A_1 = (-\alpha)^{2A-1}$, then to find the determinant of A_2 change $R'_2 = R_1 + R_2$, $R'_3 = R_3 + R'_2$, $R'_4 = R_4 + R'_3$, $R'_5 = R_5 + R'_4$ ...., $R'_A = R_A + R'_{(A-1)}$
then,

$$A_2 = \begin{pmatrix} \alpha & \cdots & \alpha & \alpha & A & 1 & \cdots & 1 & 1 \\ 0 & \cdots & \alpha & \alpha & 2A & 2 & \cdots & 2 & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & (A-1)A & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & A^2 & A & \cdots & A & A \\ 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & & & & & & & & \vdots \end{pmatrix}$$

$$\begin{array}{ccccccccc} . & \ddots & . & . & . & . & \ddots & . & \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{array}_{(2A-1)\times(2A-1)}$$

then $|A_2| = \alpha^{(A-1)}.A^2. - \alpha^{(A-1)} \Longrightarrow |A_2| = (-1)^{A-1}.A^2.\alpha^{(2A-2)}$. By substituting A_1 and A_2 to the Eq (5) we get

$$|A - \alpha I| = (-1)^{1+1}A_1 + (-1)^{A+1+1}A_2 = 0$$

$$|A - \alpha I| = (-1)^{1+1}(-\alpha)^{2A-1} + (-1)^{A+1+1}\{(-1)^{A-1}.A^2.\alpha^{(2A-2)}\} = 0$$

$$\Longrightarrow (-1)^{2A}\alpha^{2A} + (-1)^{A+2}A^2\alpha^{2A-2} = 0.$$

4 Basic Properties of A-Eccentric Tree:

Proposition 4.1. *If T is A-eccentric tree and if A is one of a p-eigenvalue of a sub matrix A of $M_p(T)$, then $-A$ is another p-eigenvalue of $M_p(T)$. And multiplicity of A and $-A$ is 1 each.*

Proof. Let tt be A-eccentric tree and A be a sub matrix of $M_p(T)$.

$$A = \begin{pmatrix} 0 & J \\ J^t & 0 \end{pmatrix}_{2A\times 2A}.$$

Clearly J is $A \times A$ sub matrix of A with all its entries 1. Let x be an eigenvector of $M_p(T)$ corresponding to α. Then,

$$\begin{pmatrix} 0 & J \\ J^t & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

And also one can verify that,

$$\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = -\alpha \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

Also, since T is A-eccentric tree $M_p(T)$, every 1^{st} column and $(A+1)^{th}$ column, every 2^{nd} column and $(A+2)^{th}$ column are linearly independent and so on, every i^{th} where $1 \le i \le A$ and $(i + A)^{th}$ column are linearly independent.

Clearly one linearly independent eigenvector for α produces one linearly independent eigenvector for $-\alpha$. Thus multiplicity of α and $-\alpha$ is one each. □

Proposition 4.2. *Suppose $A = 1$ then $M_p(T)$ has at least one positive p-eigenvalue α whose value is 1 and there is an eigenvector $y \neq 0$ such that $M_p(T)y = \alpha y$ for $\alpha > 0$.*

Proof. Suppose T is A-eccentric tree and $A = 1$. Then clearly T has exactly two peripheral vertices. Hence, $M_p(T)$ is as follows:

$$M_p(T) = \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ . & . & . & . & \ddots & . \\ . & . & . & . & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right)_{n\times n}$$

Clearly a sub matrix A of $M_p(T)$ is 2×2 matrix

$$i.e., A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2\times 2}$$

The characteristic polynomial of A is $\alpha^2 - 1 = 0$ whose p-eigenvalues are $\alpha_1 = +1$ and $\alpha_2 = -1$. Clearly α_1 and α_2 are two distinct real p-eigenvalues and one of the p-eigenvalue is 1. Thus $M_p(T)$ has at least one positive p-eigenvalue α and there is an eigenvector $y f = 0$ such that $Ay = \alpha y$. □

Observation 4.3. *If T is a tree with k peripheral vertices then T is A-eccentric if and only if $k = k_1 \cup k_2$ such that $|k_1| = |k_2|$.*

Observation 4.4. *Suppose a tree T with k peripheral vertices is A-eccentric then k is even.*

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