On $\ell$-eccentric Graphs

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Abstract

Motivated by the concept called unique eccentric point (u.e.p) [14], Kishori et al. in [2] generalized the concept as $k$-eccentric point graph. A graph $G$ is called an unique eccentric point (u.e.p) graph if each point of $G$ has a unique eccentric point where as in $k$-eccentric point graph every vertex has exactly $k$-eccentric vertices. Here we are denoting $k$-eccentric graph as $\ell$-eccentric graph and studied its property with peripheral path matrix.

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1 Introduction

Let $G$ be a connected, nontrivial graph with vertex set $V(G)$ and edge set $E(G)$ and let $|V(G)| = n$ and $|E(G)| = m$. Let $u$ and $v$ be two vertices of a graph $G$. The *distance* $d(u,v|G)$ between the vertices $u$ and $v$ is the length of a shortest path connecting $u$ and $v$. If $u = v$ then $d(u,v|G) = 0$. The *eccentricity* $e(v)$ of a vertex $v$ in a graph $G$ is the distance between $v$ and a vertex farthest from $v$ in $G$. The *diameter* $diam(G)$ of $G$ is the maximum eccentricity of $G$, while the *radius* $rad(G)$ is the smallest eccentricity of $G$. A vertex $v$ with $e(v) = diam(G)$ is called a *peripheral* vertex of $G$. A set of peripheral vertices of $G$ is called as periphery and is denoted as $P(G)$. The *peripheral path matrix* $M_p(G) = [p_{ij}]$, whose entries are 1 if there is a peripheral path between $v_i$ and $v_j$ in $G$ and 0 otherwise.

The peripheral path energy ($p$-energy (in short)) of a graph $G$ is defined as the sum of the absolute values of $p$-eigenvalues of the $M_p(G)$. i.e,

$$E_p = E_p(G) = \sum_{i=1}^{n} |\alpha_i|$$

(1)

where, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the ordinary eigenvalues [9], i.e the eigenvalues of the adjacency matrix $A(G)$. Observe that the graph energy $E(G)$ in a past few years has been extensively studied and surveyed in Mathematics and Chemistry [12, 13, 15, 17, 18, 19, 21, 23, 26]. Through out the paper $|P(G)| = k$ with labellings $v_1, v_2, \ldots, v_k$, where $2 \leq k \leq n$.

The *characteristic polynomial* of $M_p(G)$ is the $det(\alpha I - M_p(G))$. It is referred to as a characteristic polynomial of $G$ and is denoted by $\psi(G; \alpha) = c_0\alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + \ldots + c_n$. The roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the polynomial $\psi(G; \alpha)$ is called the *eigenvalues* of $M_p(G)$.

The eigenvalues of $M_p(G)$ are said to be the *peripheral path eigenvalues* (or $p$-eigenvalues (in short)) of $G$. Since $M_p(G)$ is a real, symmetric matrix, the $p$-eigenvalues are real and can be ordered in non-increasing order, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. Then the $p$-spectrum of a graph $G$ is the set of $p$-eigenvalues of $M_p(G)$, together with the multiplicities as $p$-eigenvectors of $M_p(G)$. If the $p$-eigenvalues of $M_p(G)$ are $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ and their multiplicities are $m(\alpha_1), m(\alpha_2), \ldots, m(\alpha_n)$, then the result will be

$$P - Spec(G) = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ m(\alpha_1) & m(\alpha_2) & \ldots & m(\alpha_n) \end{pmatrix}.$$

For Example, let $G$ be a graph as shown below Figure[1].
Figure 1: $G$ is a graph of order $n = 6$ with $k = 3$ peripheral vertices.

Let $\varepsilon_v = \{u : d(u, v) = diam(G)\}$. Clearly, $|\varepsilon_{v_1}| = 2$ and $|\varepsilon_{v_2}| = |\varepsilon_{v_3}| = 1$ and characteristic polynomial of $G$ is $\psi(G; \alpha) = -2\alpha^4 + \alpha^6$ whose, $p$- eigenvalues are $-\sqrt{2}, \sqrt{2}, 0, 0, 0, 0$. And hence, $p$-energy of $G$ is $2.8284$.

Motivated by the concept called unique eccentric point (u.e.p) [14], Kishori et al. in [2] generalized the concept as $k$- eccentric point graph. A graph $G$ is called an unique eccentric point (u.e.p) graph if each point of $G$ has a unique eccentric point where as in $k$-eccentric point graph every vertex has exactly $k$-eccentric vertices. Here we are denoting $k$-eccentric graph as $\ell$-eccentric graph and studied its property with peripheral path matrix. For more details about the Peripheral path matrix, Peripheral path energy, Peripheral distance energy, Peripheral path equi-energy and Peripheral Wiener index one can refer [22],

**Definition 1.1.** A vertex $u$ is said to be an eccentric vertex of $v$ if $e(v) = d(u, v)$ $u, v \in V(G)$. A graph $G$ is $\ell$- eccentric graph if for every $x$ of $V(G)$, there are $\ell$- eccentric vertices.
For Example: Refer the Figure 2.

Figure 2: $G_1, G_2$, and $G_3$ are the 1, 2 and 3 eccentric graphs respectively

2 Properties of $\ell$-eccentric graph

Theorem 2.1. Let $G$ be $\ell$-eccentric graph. Then,

1. $\ell$ is an $p$-eigenvalues of $G$.

2. If $G$ is connected, then the multiplicity of $\ell$ is 1.

3. For any $p$-eigenvalue $\alpha$ of $G$, we have $|\alpha| \leq \ell$

Proof.

1. Let $u = [1, 1, \ldots, 1]^t$, then if $M_p$ is peripheral path matrix of $G$, we have,

$$M_p = \ell u. \quad (2)$$

This is true because there are $\ell 1'$ in each row. Thus $\ell$ is a $p$-eigenvalue of $G$.

2. Let $X = [x_1, x_2, \ldots, x_n]^t$ denote any non zero vector for which

$$AX = \ell X \quad (3)$$

suppose that $x_j$ is an entry of $X$ with the largest absolute value, then the Eq (3) can be expressed as

$$(Ax)_j = \ell x_j$$

and hence

$$\sum x_i = \ell x_j$$

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where summation runs over those ℓ vertices \( v_i \) which are an eccentric vertex to \( v_j \). If \( G \) is connected, we may proceed successively in this way, eventually showing that all entries of \( X \) are equal. Thus \( X \) is a multiple of \( u \) and the space of eigenvector associated with the p-eigenvalue \( ℓ \) has dimension 1.

3. Suppose that

\[
Ay = \alpha y , \ y \neq 0
\]

and \( y_j \) denote an entry of \( y \) which is least in absolute value, as in condition 2. we have,

\[
\sum y_i = \alpha y_j \quad \text{and} \\
|\alpha||y_j| = |\sum y_i| \\
|\alpha||y_j| \leq \sum |y_i| \\
|\alpha||y_j| = ℓ|y_i| \\
\Rightarrow |\alpha| \leq ℓ \quad \text{as required.}
\]

Proposition 2.2. If \( G \) is ℓ-eccentric graph and \( \alpha \) is a p-eigenvalue of \( M_p(G) \), then no p-eigenvalue of \( M_p(G) \) has absolute value greater than \( ℓ \).

Proof. Let \( \alpha \) be a p-eigenvalue of \( M_p(G) \) and \( x = [x_1, x_2, \ldots, x_k]^t \) be corresponding eigenvector. Let \( x_i \) be the entry of \( x \) whose absolute value is greater. Hence,

\[
\alpha x_i = \sum_{j=1}^{k} a_{ij} x_j ; i = 1, 2, \ldots k.
\]

Now

\[
|\alpha||x_i| \leq |x_i| \sum_{j=1}^{k} a_{ij} ; i = 1, 2, \ldots k.
\]

since \( G \) is ℓ– eccentric graph,

\[
\sum_{j=1}^{k} a_{ij} = ℓ ; i = 1, 2, \ldots k.
\]

Hence,

\[
|\alpha||x_i| \leq |x_i|.|\ell| \quad \Rightarrow \quad |\alpha| \leq |\ell|.
\]
**Corollary 2.3.** Suppose $G$ is unique eccentric point graph then no p-eigenvalue of $M_p(G)$ has absolute value greater than 1.

*Proof.* Let $\alpha$ be a p-eigenvalue of $M_p(G)$ and $x = [x_1, x_2, ... x_k]^t$ corresponding eigenvector. Let $x_i$ be the entry of $x$ whose absolute value is greatest. Hence,

$$\alpha x_i = \sum_{j=1}^{k} a_{ij} x_j ; i = 1, 2, ... k.$$  

Now,

$$|\alpha||x_i| \leq |x_i| \sum_{j=1}^{k} a_{ij} ; i = 1, 2, ... k.$$  

since $G$ is unique eccentric point graph,

$$\implies |\alpha| \leq 1.$$  

**Proposition 2.4.** Let $M_p(G)$ be $n \times n$ real matrix with $v_1, v_2, ... v_k$ peripheral vertices. Let $M_{p_1}(G)$ be $k \times k$ real sub matrix of $M_p(G)$. If $G$ is unique eccentric point graph then there is a non-zero column vector $x$ such that $Ax = x$

*Proof.* Let $M_{p_1}(G) - I$ be a matrix which has the properties that the sum of the entries in each column is equal to zero. Since $G$ is unique eccentric point graph, $M_{p_1}(G) - I$ results in to a matrix whose first row is zero. (If not a small operation can be applied, i.e. $R_1 \rightarrow R_1 + R_2 +, ..., + R_k$ to $M_{p_1}(G) - I$). Hence, $|M_{p_1}(G) - I| = 0$. Thus 1 is a p-eigenvalue of $A$ and there is an eigenvector $x \neq 0$ such that $Ax = x$.  

3. $\ell$-Eccentric Trees:

For $\ell$-eccentric tree, the peripheral path matrix $M_p(T)$ is as follows:
This matrix representation takes the form

\[
M_p(T) = \begin{bmatrix}
A_{2\ell \times 2\ell} & B_{2\ell \times (n-2\ell)} \\
B^t_{(n-2\ell) \times 2\ell} & C_{(n-2\ell) \times (n-2\ell)}
\end{bmatrix}
\]

Note that sub matrix \(B, B^t\) and \(C\) are zero - matrix, where as \(A\) is non-zero matrix. A sub matrix \(A\) is \(2\ell \times 2\ell\) sub matrix of \(M_p(T)\), whose entries are as follows:

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

Symbolically \(A\) can be represented as

\[
A = \begin{bmatrix}
0_{\ell \times \ell} & J_{\ell \times \ell} \\
J^t_{\ell \times \ell} & 0_{\ell \times \ell}
\end{bmatrix}_{2\ell \times 2\ell}
\]

Where \(J\) is a \(\ell \times \ell\) matrix having all entries 1.

**Observation 3.1.** A sub matrix \(A\) of \(M_p(T)\) has just two linearly independent rows and so its rank is 2. Consequently 0 is an \(p\)-eigenvalue of \(M_p(T)\) with multiplicity \(2\ell - 2\).
Next, we give the Characteristic polynomial of a sub matrix $A$ of $M_p(T)$. Suppose,

$$A = \begin{bmatrix}
  a_{11} & \cdots & a_{1,(\ell-1)} & a_{1,\ell} \\
  a_{21} & \cdots & a_{2,(\ell-1)} & a_{2,\ell} \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{(\ell-1),1} & \cdots & a_{(\ell-1),(\ell-1)} & a_{(\ell-1),\ell} \\
  a_{\ell,1} & \cdots & a_{\ell,(\ell-1)} & a_{\ell,\ell}
\end{bmatrix}$$

Then, the Characteristic polynomial of $A$ is

$$\begin{bmatrix}
  a_{11} & \cdots & a_{1,(\ell-1)} & a_{1,\ell} \\
  a_{21} & \cdots & a_{2,(\ell-1)} & a_{2,\ell} \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{(\ell-1),1} & \cdots & a_{(\ell-1),(\ell-1)} & a_{(\ell-1),\ell} \\
  a_{\ell,1} & \cdots & a_{\ell,(\ell-1)} & a_{\ell,\ell}
\end{bmatrix} - \alpha I = 0$$

Thus, the Characteristic polynomial of $A$ is

$$\begin{bmatrix}
  0 & \cdots & 0 & 0 \\
  0 & \cdots & 0 & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & 0 & 0 \\
  0 & \cdots & 0 & 0 \\
\end{bmatrix}$$

Then, subtract $(\ell + 1)^{th}$ row from $(\ell + 2)^{th}$, $(\ell + 3)^{th}$, ..., $(2\ell - 1)^{th}$, $(2\ell)^{th}$ rows, then

$$|A - \alpha I| = \begin{bmatrix}
  -\alpha & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
  0 & -\alpha & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & -\alpha & 0 & 1 & 1 & \cdots & 1 & 1 \\
  0 & 0 & \cdots & 0 & -\alpha & 1 & 1 & \cdots & 1 & 1 \\
  1 & 1 & \cdots & 1 & 1 & -\alpha & 0 & \cdots & 0 & 0 \\
  1 & 1 & \cdots & 1 & 1 & 0 & -\alpha & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & -\alpha & 0 \\
  1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & -\alpha
\end{bmatrix}_{2\ell \times 2\ell} = 0$$

Now, subtract $(\ell + 1)^{th}$ row from $(\ell + 2)^{th}$, $(\ell + 3)^{th}$, ..., $(2\ell - 1)^{th}$, $(2\ell)^{th}$ rows, then
the result will be,

\[
\begin{bmatrix}
-\alpha & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & -\alpha & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\alpha & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & -\alpha & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & -\alpha & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \alpha & -\alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \alpha & 0 & \cdots & -\alpha & 0 \\
0 & 0 & \cdots & 0 & 0 & \alpha & 0 & \cdots & 0 & -\alpha
\end{bmatrix}_{2\ell \times 2\ell} = 0
\]

Adding \((\ell + 2)^{th}\), \((\ell + 3)^{th}\), ..., \((2\ell - 1)^{th}\), \((2\ell)^{th}\) column to \((\ell + 1)^{th}\) column,

\[
\begin{bmatrix}
-\alpha & 0 & \cdots & 0 & 0 & \ell & 1 & \cdots & 1 & 1 \\
0 & -\alpha & \cdots & 0 & 0 & \ell & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\alpha & 0 & \ell & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & -\alpha & \ell & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & -\alpha & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha
\end{bmatrix}_{2\ell \times 2\ell} = 0
\]

Again by subtracting \((1)^{st}\) column from \((2)^{nd}\), \((3)^{rd}\), ..., \((\ell)^{th}\) column, we have

\[
\begin{bmatrix}
-\alpha & \alpha & \cdots & \alpha & \alpha & \ell & 1 & \cdots & 1 & 1 \\
0 & -\alpha & \cdots & 0 & 0 & \ell & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\alpha & 0 & \ell & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & -\alpha & \ell & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 0 & 0 & -\alpha & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha
\end{bmatrix}_{2\ell \times 2\ell} = 0
\]

Expanding from \(1^{st}\) column.

\[
\implies (-1)^{1+1}A_1 + (-1)^{\ell+1+1}A_2 = 0 \tag{5}
\]
Let $A_1 = \begin{vmatrix} -\alpha & \cdots & 0 & 0 & \ell & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\alpha & 0 & \ell & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & -\alpha & \ell & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & -\alpha & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{vmatrix}_{(2\ell-1)\times(2\ell-1)}$

Let $A_2 = \begin{vmatrix} \alpha & \cdots & \alpha & \alpha & \ell & 1 & \cdots & 1 & 1 \\ -\alpha & \cdots & 0 & 0 & \ell & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\alpha & 0 & \ell & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & -\alpha & \ell & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{vmatrix}_{(2\ell-1)\times(2\ell-1)}$

now, the determinant of $A_1 = (-\alpha)^{2\ell-1}$, then to find the determinant of $A_2$ change $R'_2 = R_1 + R_2, R'_3 = R_3 + R'_2, R'_4 = R_4 + R'_3, R'_5 = R_5 + R'_4, \ldots, R'_\ell = R_\ell + R'_{(\ell-1)}$

then,

$A_2 = \begin{vmatrix} \alpha & \cdots & \alpha & \alpha & \ell & 1 & \cdots & 1 & 1 \\ 0 & \cdots & \alpha & \alpha & 2\ell & 2 & \cdots & 2 & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & (\ell-1)\ell & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & \ell^2 & \ell & \cdots & \ell & \ell \\ 0 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha \end{vmatrix}_{(2\ell-1)\times(2\ell-1)}$

then $|A_2| = \alpha^{(\ell-1)}\ell^2 - \alpha^{(\ell-1)} \implies |A_2| = (-1)^{\ell-1}\ell^2\alpha^{(2\ell-2)}$. By substituting $A_1$ and $A_2$ to the Eq (5) we get

$|A - \alpha I| = (-1)^{1+1}A_1 + (-1)^{\ell+1}A_2 = 0$

$|A - \alpha I| = (-1)^{1+1}(-\alpha)^{2\ell-1} + (-1)^{\ell+1}\{(-1)^{\ell-1}\ell^2\alpha^{(2\ell-2)}\} = 0$

$\implies (-1)^{2\ell}\alpha^{2\ell} + (-1)^{\ell+2}\ell^2\alpha^{2\ell-2} = 0$. 

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4 Basic Properties of $\ell$-Eccentric Tree:

**Proposition 4.1.** If $T$ is $\ell$-eccentric tree and if $\ell$ is one of a $p$-eigenvalue of a sub matrix $A$ of $M_p(T)$, then $-\ell$ is another $p$-eigenvalue of $M_p(T)$. And multiplicity of $\ell$ and $-\ell$ is 1 each.

**Proof.** Let $G$ be $\ell$-eccentric tree and $A$ be a sub matrix of $M_p(T)$.

\[ A = \begin{bmatrix} 0 & J \\ J^t & 0 \end{bmatrix}_{2\ell \times 2\ell}. \]

Clearly $J$ is $\ell \times \ell$ sub matrix of $A$ with all its entries 1. Let $x$ be an eigenvector of $M_p(T)$ corresponding to $\alpha$. Then,

\[ \begin{bmatrix} 0 & J \\ J^t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

And also one can verify that,

\[ \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = -\alpha \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \]

Also, since $T$ is $\ell$-eccentric tree $M_p(T)$, every $1^{st}$ column and $(\ell + 1)^{th}$ column, every $2^{nd}$ column and $(\ell + 2)^{th}$ column are linearly independent and so on, every $i^{th}$ where $1 \leq i \leq \ell$ and $(i + \ell)^{th}$ column are linearly independent.

Clearly one linearly independent eigenvector for $\alpha$ produces one linearly independent eigenvector for $-\alpha$. Thus multiplicity of $\alpha$ and $-\alpha$ is one each. \qed

**Proposition 4.2.** Suppose $\ell = 1$ then $M_p(T)$ has at least one positive $p$-eigenvalue $\alpha$ whose value is 1 and there is an eigenvector $y \neq 0$ such that $M_p(T)y = \alpha y$ for $\alpha > 0$.

**Proof.** Suppose $T$ is $\ell$-eccentric tree and $\ell = 1$. Then clearly $T$ has exactly two peripheral vertices. Hence, $M_p(T)$ is as follows:

\[ M_p(T) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \]

Clearly a sub matrix $A$ of $M_p(T)$ is $2 \times 2$ matrix

\[ i.e., \ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2 \times 2} \]
The characteristic polynomial of $A$ is $\alpha^2 - 1 = 0$ whose p-eigenvalues are $\alpha_1 = +1$ and $\alpha_2 = -1$. Clearly $\alpha_1$ and $\alpha_2$ are two distinct real p-eigenvalues and one of the p-eigenvalue is 1. Thus $M_p(T)$ has at least one positive p-eigenvalue $\alpha$ and there is an eigenvector $y \neq 0$ such that $Ay = \alpha y$.

**Observation 4.3.** If $T$ is a tree with $k$ peripheral vertices then $T$ is $\ell$-eccentric if and only if $k = k_1 \cup k_2$ such that $|k_1| = |k_2|$.

**Observation 4.4.** Suppose a tree $T$ with $k$ peripheral vertices is $\ell$-eccentric then $k$ is even.

**References**


