On $A$-eccentric Graphs

Kishori P. Narayankar†  Lokesh. S. B.†

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Abstract
Motivated by the concept called unique eccentric point (u.e.p) [14], Kishori et al. in [2] generalized the concept as $k$- eccentric point graph. A graph $tt$ is called an unique eccentric point (u.e.p) graph if each point of $tt$ has a unique eccentric point where as in $k$-eccentric point graph every vertex has exactly $k$-eccentric vertices. Here we are denoting $k$-eccentric graph as $A$-eccentric graph and studied its property with peripheral path matrix.

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1 Introduction

Let \( tt \) be a connected, nontrivial graph with vertex set \( V(tt) \) and edge set \( E(tt) \) and let \(|V(tt)| = n \) and \(|E(tt)| = m \). Let \( u \) and \( v \) be two vertices of a graph \( tt \). The distance \( d(u, v|tt) \) between the vertices \( u \) and \( v \) is the length of a shortest path connecting \( u \) and \( v \). If \( u = v \) then \( d(u, v|tt) = 0 \). The eccentricity \( e(v) \) of a vertex \( v \) in a graph \( tt \) is the distance between \( v \) and a vertex farthest from \( v \) in \( tt \). The diameter \( diam(tt) \) of \( tt \) is the maximum eccentricity of \( tt \), while the radius \( rad(tt) \) is the smallest eccentricity of \( tt \). A vertex \( v \) with \( e(v) = diam(tt) \) is called a peripheral vertex of \( tt \). A set of peripheral vertices of \( tt \) is called as periphery and is denoted as \( P(tt) \). The peripheral path matrix \( M_p(tt) = [p_{ij}] \), whose entries are 1 if there is a peripheral path between \( v_i \) and \( v_j \) in \( tt \) and 0 otherwise.

The peripheral path energy (p-energy (in short)) of a graph \( tt \) is defined as the sum of the absolute values of p- eigenvalues of the \( M_p(tt) \). i.e,

\[
E_p = E_p(tt) = \sum_{i=1}^{\infty} |\alpha_i|
\]  

The form of Eq. (1) is chosen so as to be fully analogous to the definition of graph energy \([8, 5, 7]\).

\[
E = E(tt) = \sum_{i=1}^{\infty} |\lambda_i|
\]

where, \( \lambda_1, \lambda_2, ..., \lambda_n \) are the ordinary eigenvalues \([9]\), i.e the eigenvalues of the adjacency matrix \( A(tt) \). Observe that the graph energy \( E(tt) \) in a past few years has been extensively studied and surveyed in Mathematics and Chemistry \([12, 13, 15, 17, 18, 19, 21, 25, 26, 10, 11, 16, 20, 23]\). Through out the paper \( |P(tt)| = k \) with labellings \( v_1, v_2, ..., v_k \), where \( 2 \leq k \leq n \).

The characteristic polynomial of \( M_p(tt) \) is the \( det(\alpha I - M_p(tt)) \). It is referred to as a characteristic polynomial of \( tt \) and is denoted by \( \psi(tt; \alpha) = c_0\alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + ... + c_n \). The roots \( \alpha_1, \alpha_2, ..., \alpha_n \) of the polynomial \( \psi(tt; \alpha) \) is called the eigenvalues of \( M_p(tt) \).

The eigenvalues of \( M_p(tt) \) are said to be the peripheral path eigenvalues (or p-eigenvalues in short) of \( tt \). Since \( M_p(tt) \) is a real, symmetric matrix, the p-eigenvalues are real and can be ordered in non-increasing order, \( \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n \). Then the p-spectrum of a graph \( tt \) is the set of p-eigenvalues of \( M_p(tt) \), together with the multiplicities as p-eigenvalues of \( M_p(tt) \). If the p-eigenvalues of \( M_p(tt) \) are \( \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n \) and their multiplicities are \( m(\alpha_1), m(\alpha_2), ..., m(\alpha_n) \), then the result will be

\[
p-spec(tt) = \begin{pmatrix}
\alpha_1 & \alpha_2 & ... & \alpha_n \\
m(\alpha_1) & m(\alpha_2) & ... & m(\alpha_n)
\end{pmatrix}
\]

For Example, let \( tt \) be a graph as shown below Figure 1.
Figure 1: $tt$ is a graph of order $n = 6$ with $k = 3$ peripheral vertices.

$M_{p}(tt) =
\begin{array}{cccccc}
\cdot & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
v_1 & 0 & 1 & 1 & 0 & 0 & 0 \\
v_2 & 1 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 1 & 0 & 0 & 0 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$

Let $\varepsilon_r = \{u : d(u, v) = diam(tt)\}$. Clearly, $|\varepsilon_{v_1}| = 2$ and $|\varepsilon_{v_2}| = |\varepsilon_{v_3}| = 1$ and characteristic polynomial of $tt$ is $\psi(tt; \alpha) = -2\alpha^4 + \alpha^6$ whose, $p$- eigenvalues are $-2, 2, 0, 0, 0, 0$. And hence, $p$-energy of $tt$ is 2.8284.

Motivated by the concept called unique eccentric point (u.e.p) [14], Kishori et al. in [2] generalized the concept as $k$-eccentric point graph. A graph $tt$ is called an unique eccentric point (u.e.p) graph if each point of $tt$ has a unique eccentric point where as in $k$-eccentric point graph every vertex has exactly $k$-eccentric vertices. Here we are denoting $k$-eccentric graph as $A$-eccentric graph and studied its property with peripheral path matrix. For more details about the Peripheral path matrix, Peripheral path energy, Peripheral distance energy, Peripheral path equi-energy and Peripheral Wiener index one can refer [22],

**Definition 1.1.** A vertex $u$ is said to be an eccentric vertex of $v$ if $e(v) = d(u, v)$ $u, v \in V(tt)$. A graph $tt$ is $A$- eccentric graph if for every $x$ of $V(tt)$, there are $A$-eccentric vertices.
For Example: Refer the Figure 2.

Figure 2: $tt_1$, $tt_2$, and $tt_3$ are the 1, 2 and 3 eccentric graphs respectively

## 2 Properties of $A$-eccentric graph

**Theorem 2.1.** Let $tt$ be $A$-eccentric graph. Then,

1. $A$ is an $p$-eigenvalues of $tt$.
2. If $tt$ is connected, then the multiplicity of $A$ is 1.
3. For any $p$-eigenvalue $\alpha$ of $tt$, we have $|\alpha| \leq A$

**Proof.**

1. Let $u = [1, 1, 1, ..., 1]^T$, then if $M_p$ is peripheral path matrix of $tt$, we have,

$$M_p = Au.$$  \hspace{1cm} (2)

This is true because there are $A 1^T$ in each row. Thus $A$ is a $p$-eigenvalue of $tt$.

2. Let $X = [x_1, x_2, ..., x_n]^T$ denote any non zero vector for which

$$AX = AX.$$  \hspace{1cm} (3)

suppose that $x_j$ is an entry of $X$ with the largest absolute value, then the Eq (3) can be expressed as

$$(Ax)_j = Ax_j$$

and hence

$$\sum x_i = Ax_j$$
where summation runs over those \( A \) vertices \( v_i \) which are an eccentric vertex to \( v_j \). If \( tt \) is connected, we may proceed successively in this way, eventually showing that all entries of \( X \) are equal. Thus \( X \) is a multiple of \( u \) and the space of eigenvector associated with the \( p \)-eigenvalue \( A \) has dimension 1.

3. Suppose that
\[
Ay = ay \quad , \quad y \neq 0
\]
and \( y_i \) denote an entry of \( y \) which is least in absolute value, as in condition 2. we have,
\[
\sum y_i = ay_i \quad \text{and} \quad |a||y_j| = \sum |y_i|
\]
\[
|a||y_j| \leq \sum |y_i|
\]
\[
|a||y_j| = A|y_i|
\]
\[
\Rightarrow |a| \leq A \quad \text{as required.}
\]

Proposition 2.2. If \( tt \) is \( A \)-eccentric graph and \( \alpha \) is a \( p \)-eigenvalue of \( M_p(tt) \), then no \( p \)-eigenvalue of \( M_p(tt) \) has absolute value greater than \( A \).

Proof. Let \( \alpha \) be a \( p \)-eigenvalue of \( M_p(tt) \) and \( x = [x_1, x_2, ..., x_k]^t \) be corresponding eigenvector. Let \( x_i \) be the entry of \( x \) whose absolute value is greater. Hence,
\[
\alpha x_i = \sum_{j=1}^{k} a_{ij} x_j \quad ; \quad i = 1, 2, ...k.
\]

Now
\[
|\alpha||x_i| \leq |x_i| \sum_{j=1}^{k} a_{ij} ; \quad i = 1, 2, ...k.
\]

since \( tt \) is \( A \)- eccentric graph,
\[
\sum_{j=1}^{k} a_{ij} = A \quad ; \quad i = 1, 2, ...k.
\]

Hence,
\[
|\alpha||x_i| \leq |x_i||A| \Rightarrow |\alpha| \leq |A|.
\]

\[ \square \]
**Corollary 2.3.** Suppose \( tt \) is unique eccentric point graph then no p-eigenvalue of \( M_p(tt) \) has absolute value greater than 1.

*Proof.* Let \( \alpha \) be a p-eigenvalue of \( M_p(tt) \) and \( x = [x_1, x_2, ... x_k]' \) corresponding eigenvector. Let \( x_i \) be the entry of \( x \) whose absolute value is greatest. Hence,

\[
\alpha x_i = \sum_{j=1}^{k} a_{ij} x_j \ ; \ i = 1, 2, ... k.
\]

Now,

\[
|\alpha| |x_i| \leq \sum_{j=1}^{k} a_{ij} \ ; \ i = 1, 2, ... k.
\]

since \( tt \) is unique eccentric point graph,

\[
\Rightarrow |\alpha| \leq 1.
\]

\[\Box\]

**Proposition 2.4.** Let \( M_p(tt) \) be \( n \times n \) real matrix with \( v_1, v_2, ..., v_k \) peripheral vertices. Let \( M_{p1}(tt) \) be \( k \times k \) real sub matrix of \( M_p(tt) \). If \( tt \) is unique eccentric point graph then there is a non-zero column vector \( x \) such that \( Ax = x \)

*Proof.* Let \( M_{p1}(tt) - I \) be a matrix which has the properties that the sum of the entries in each column is equal to zero. Since \( tt \) is unique eccentric point graph, \( M_{p1}(tt) - I \) results in a matrix whose first row is zero. (If not a small operation can be applied, i.e. \( R_1 \rightarrow R_1+R_2+, ..., +R_k \) to \( M_{p1}(tt)-I \). Hence, \( |M_{p1}(tt)-I| = 0 \). Thus 1 is a p-eigenvalue of \( A \) and there is an eigenvector \( x \neq 0 \) such that \( Ax = x \).

\[\Box\]

### 3 A-Eccentric Trees:

For A-eccentric tree, the peripheral path matrix \( M_p(T) \) is as follows:
and so its Observation

Where

Symbolically

A sub matrix

Note that sub matrix

This matrix representation takes the form

\[
M_p(T) = \frac{\Sigma}{B_{(n-2A)\times 2A}} \frac{B_{2A\times(n-2A)}}{C_{(n-2A)\times n-2A}}. 
\]

Note that sub matrix \( B, B' \) and \( C \) are zero - matrix, where as \( A \) is non-zero matrix.

A sub matrix \( A \) is \( 2A \times 2A \) sub matrix of \( M_p(T) \), whose entries are as follows:

\[
A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 
\end{pmatrix}
\]

Symbolically \( A \) can be represented as

\[
A = \frac{\Sigma}{J_{A\times A}} \frac{J_{A\times A}}{0_{A\times 2A}}. 
\]

Where \( J \) is a \( A \times A \) matrix having all entries 1.

**Observation.** A sub matrix \( A \) of \( M_p(T) \) has just two linearly independent rows and so its rank is 2. Consequently \( 0 \) is an \( p \)-eigenvalue of \( M_p(T) \) with multiplicity \( 2A - 2 \).
Next, we give the Characteristic polynomial of a sub matrix $A$ of $M_p(T)$. Suppose,

$$A = \begin{array}{cccc}
  a_{11} & \ldots & a_{1,(A-1)} & a_{1,A} \\
  a_{21} & \ldots & a_{2,(A-1)} & a_{2,A} \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{(A-1),1} & \ldots & a_{(A-1),(A-1)} & a_{(A-1),A} \\
  a_{A,1} & \ldots & a_{A,(A-1)} & a_{A,A} \\
  a_{(A+1),1} & \ldots & a_{(A+1),(A-1)} & a_{(A+1),A} \\
  a_{(A+2),1} & \ldots & a_{(A+2),(A-1)} & a_{(A+2),A} \\
  a_{(2A-1),1} & \ldots & a_{(2A-1),(A-1)} & a_{(2A-1),A} \\
  a_{(2A),1} & \ldots & a_{(2A),(A-1)} & a_{(2A),A} \\
\end{array}$$

$$\Rightarrow A = \begin{array}{cccc}
  0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 0 \\
  1 & 1 & \cdots & 1 \\
  1 & 1 & \cdots & 1 \\
  \vdots & \ddots & \vdots & \vdots \\
  1 & 1 & \cdots & 1 \\
\end{array}$$

then,

$$\begin{array}{cccc}
  -a & 0 & \cdots & 0 \\
  \vdots & -a & \cdots & 0 \\
  0 & 0 & \cdots & -a \\
  0 & 0 & \cdots & 1 \\
  1 & 1 & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \cdots & 1 \\
\end{array}$$

$$|A - aI| = \begin{array}{cccc}
  1 & 1 & \cdots & 1 \\
  1 & 1 & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \cdots & 1 \\
\end{array}$$

now, subtract $(A + 1)^{th}$ row from $(A + 2)^{th}$, $(A + 3)^{th}$, ..., $(2A - 1)^{th}$, $(2A)^{th}$ rows, then
the result will be,

\[
\begin{pmatrix}
-a & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & -a & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & -a & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & -a & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & -a & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & a & -a & \cdots & 0 & 0 \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & a & 0 & \cdots & -a & 0 \\
0 & 0 & \cdots & 0 & 0 & a & 0 & \cdots & 0 & -a \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -a & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -a \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
= 0
\]

Adding \((A + 2)^{th}\), \((A + 3)^{th}\), ..., \((2A - 1)^{th}\), \((2A)^{th}\) column to \((A + 1)^{th}\) column,

\[
\begin{pmatrix}
-a & 0 & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\
0 & -a & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & -a & 0 & A & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & -a & A & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & -a & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -a & \cdots & 0 & 0 \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -a & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -a \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
= 0
\]

Again by subtracting \((1)^{st}\) column from \((2)^{nd}\), \((3)^{rd}\), ..., \((A)^{th}\) column, we have

\[
\begin{pmatrix}
-a & a & \cdots & a & a & A & 1 & \cdots & 1 & 1 \\
0 & -a & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\
& & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]
Expanding from $1^{st}$ column.

$$\Rightarrow (-1)^{1+1}A_1 + (-1)^{4+1+1}A_2 = 0$$

(5)
\[
\begin{align*}
&\begin{array}{cccccccc}
-a & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & -a & -a & A & 1 & \cdots & 1 & 1 \\
0 & \cdots & 0 & 0 & -a & 0 & \cdots & 1 & 1 \\
0 & \cdots & 0 & 0 & 0 & -a & \cdots & 0 & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -a \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -a \\
\end{array}
\end{align*}
\]

Let \( A_1 = \begin{array}{cccccccc}
-a & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & -a & -a & A & 1 & \cdots & 1 & 1 \\
0 & \cdots & 0 & 0 & -a & 0 & \cdots & 1 & 1 \\
0 & \cdots & 0 & 0 & 0 & -a & \cdots & 0 & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -a \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -a \\
\end{array} (2A-1) \times (2A-1)
\]

\[A_2 = \begin{array}{cccccccc}
-a & \cdots & 0 & 0 & A & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & -a & 0 & A & 1 & \cdots & 1 & 1 \\
0 & \cdots & 0 & -a & A & 1 & \cdots & 1 & 1 \\
0 & \cdots & 0 & 0 & -a & 0 & \cdots & 0 & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & -a & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -a \\
\end{array} (2A-1) \times (2A-1)
\]

Now, the determinant of \( A_3 = (-a)^{2A-1} \), then to find the determinant of \( A_2 \) change \( \begin{array}{c}
R_2' = R_1 + R_2, \\
R_3' = R_3 + R', \\
R_4' = R_4 + R', \\
R_5' = R_5 + R' \ldots, \\
R_A' = R_A + R'
\end{array} (A-1) \)

Then,
\[
\begin{align*}
&\begin{array}{cccccccc}
a & \cdots & a & a & A & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & a & a & 2A & 2 & \cdots & 2 & 2 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & (A-1)A & 1 & \cdots & 1 & 1 \\
\end{array} (A-1)
\end{align*}
\]

\[
A_2 = \begin{array}{cccccccc}
0 & \cdots & 0 & 0 & (A-1)A & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & A^2 & A & \cdots & A & A \\
0 & \cdots & 0 & 0 & 0 & -a & \cdots & 0 & 0 \\
\end{array}
\]
then $|A_2| = a^{(A-1)}A^2 - a^{(A-1)} \Rightarrow |A_2| = (-1)^{A-1}A^2a^{(2A-2)}$. By substituting $A_1$ and $A_2$ to the Eq (5) we get

$$|A - \alpha I| = (-1)^{A+1}A_1 + (-1)^{A+1+1}A_2 = 0$$

$$|A - \alpha I| = (-1)^{A+1}(-\alpha)^{2A-1} + (-1)^{A+1+1}[(-1)^{A-1}A^2a^{(2A-2)}] = 0$$

$$\Rightarrow (-1)^{2A}a^{2A} + (-1)^{A+2}A^2a^{2A-2} = 0.$$
4 Basic Properties of $A$-Eccentric Tree:

**Proposition 4.1.** If $T$ is $A$-eccentric tree and if $A$ is one of a $p$-eigenvalue of a sub matrix $A$ of $M_p(T)$, then $-A$ is another $p$-eigenvalue of $M_p(T)$. And multiplicity of $A$ and $-A$ is 1 each.

*Proof.* Let $tt$ be $A$-eccentric tree and $A$ be a sub matrix of $M_p(T)$.

$$A = \begin{bmatrix} J & \Sigma J \\ \Sigma J & 0 \end{bmatrix}.$$

Clearly $J$ is $A \times A$ sub matrix of $A$ with all its entries 1. Let $x$ be an eigenvector of $M_p(T)$ corresponding to $\alpha$. Then,

$$\begin{bmatrix} J & \Sigma J \\ \Sigma J & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

And also one can verify that,

$$\begin{bmatrix} J & \Sigma J \\ \Sigma J & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}.$$

Also, since $T$ is $A$-eccentric tree $M_p(T)$, every $i^{th}$ column and $(A+1)^{th}$ column, every $2^{nd}$ column and $(A+2)^{th}$ column are linearly independent and so on, every $i^{th}$ where $1 \leq i \leq A$ and $(i + A)^{th}$ column are linearly independent.

Clearly one linearly independent eigenvector for $\alpha$ produces one linearly independent eigenvector for $-\alpha$. Thus multiplicity of $\alpha$ and $-\alpha$ is one each. \hfill $\square$

**Proposition 4.2.** Suppose $A = 1$ then $M_p(T)$ has at least one positive $p$-eigenvalue $\alpha$ whose value is 1 and there is an eigenvector $y \neq 0$ such that $M_p(T)y = \alpha y \neq 0$, $\alpha > 0$.

*Proof.* Suppose $T$ is $A$-eccentric tree and $A = 1$. Then clearly $T$ has exactly two peripheral vertices. Hence, $M_p(T)$ is as follows:

$$M_p(T) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}.$$

Clearly a sub matrix $A$ of $M_p(T)$ is $2 \times 2$ matrix

$$i.e., A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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The characteristic polynomial of $A$ is $\alpha^2 - 1 = 0$ whose $p$-eigenvalues are $\alpha_1 = +1$ and $\alpha_2 = -1$. Clearly $\alpha_1$ and $\alpha_2$ are two distinct real $p$-eigenvalues and one of the $p$-eigenvalue is 1. Thus $M_p(T)$ has at least one positive $p$-eigenvalue $\alpha$ and there is an eigenvector $y \neq 0$ such that $Ay = \alpha y$.

\textbf{Observation 4.3.} If $T$ is a tree with $k$ peripheral vertices then $T$ is $A$-eccentric if and only if $k = k_1 \cup k_2$ such that $|k_1| = |k_2|$.

\textbf{Observation 4.4.} Suppose a tree $T$ with $k$ peripheral vertices is $A$-eccentric then $k$ is even.

\textbf{References}


